Math 1433

15 January 2024

Warm-up:

List all angles

Exponential form

Last time

Fact: $e^{\theta i} = \cos(\theta) + \sin(\theta)\, i$

Multiplying both sides of this by r gives

$$r\, e^{\theta i} = r\cos(\theta) + r\sin(\theta)\, i.$$

Writing a complex number as $\_\_ e^{\_\_ i}$ is called **exponential form**.

Example: $4\sqrt{3} + 4i \quad = \quad 8\cos(\frac{\pi}{6}) + 8\sin(\frac{\pi}{6})\, i \quad = \quad 8e^{(\pi/6)i}$

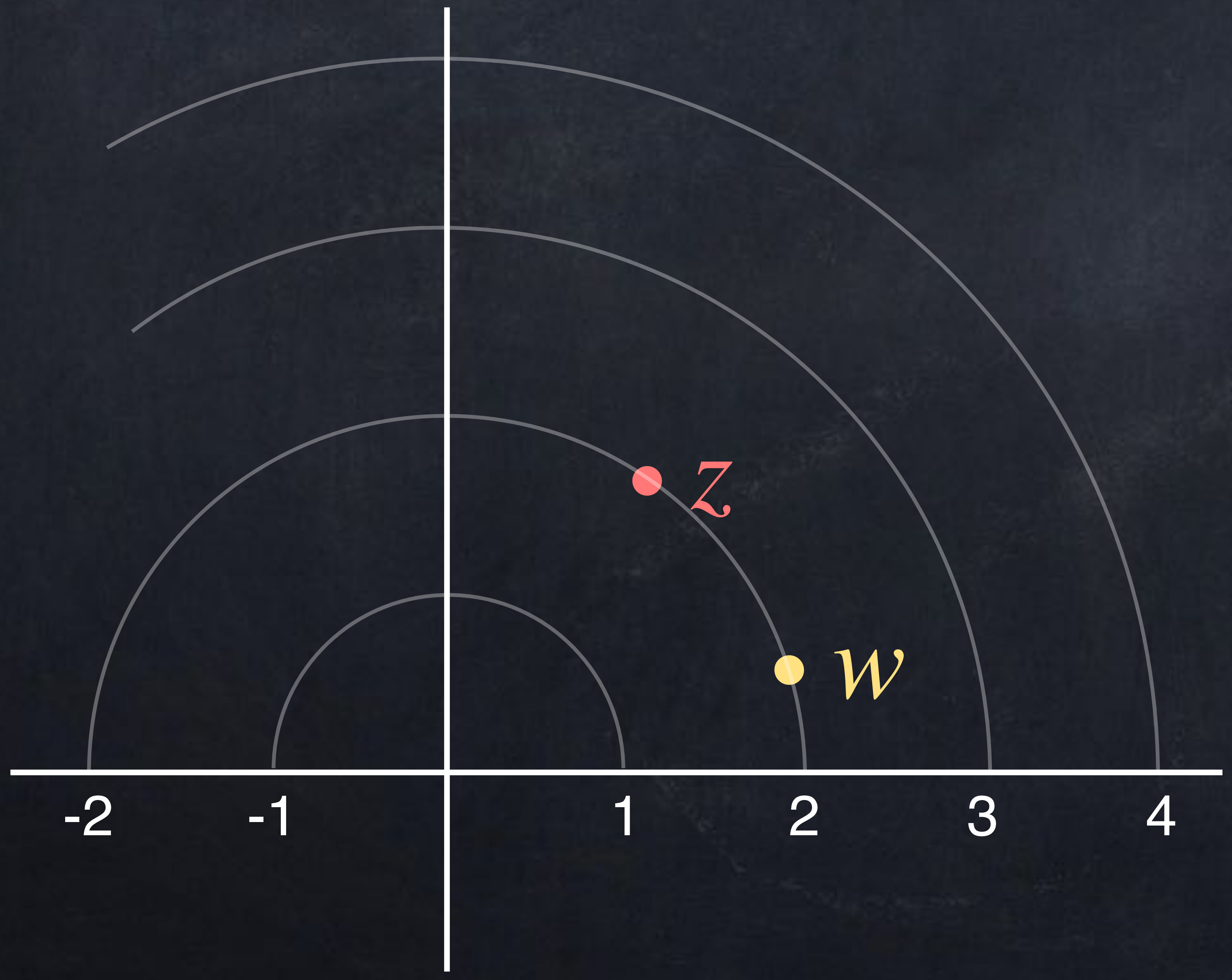
rectangular form — polar form — exponential form

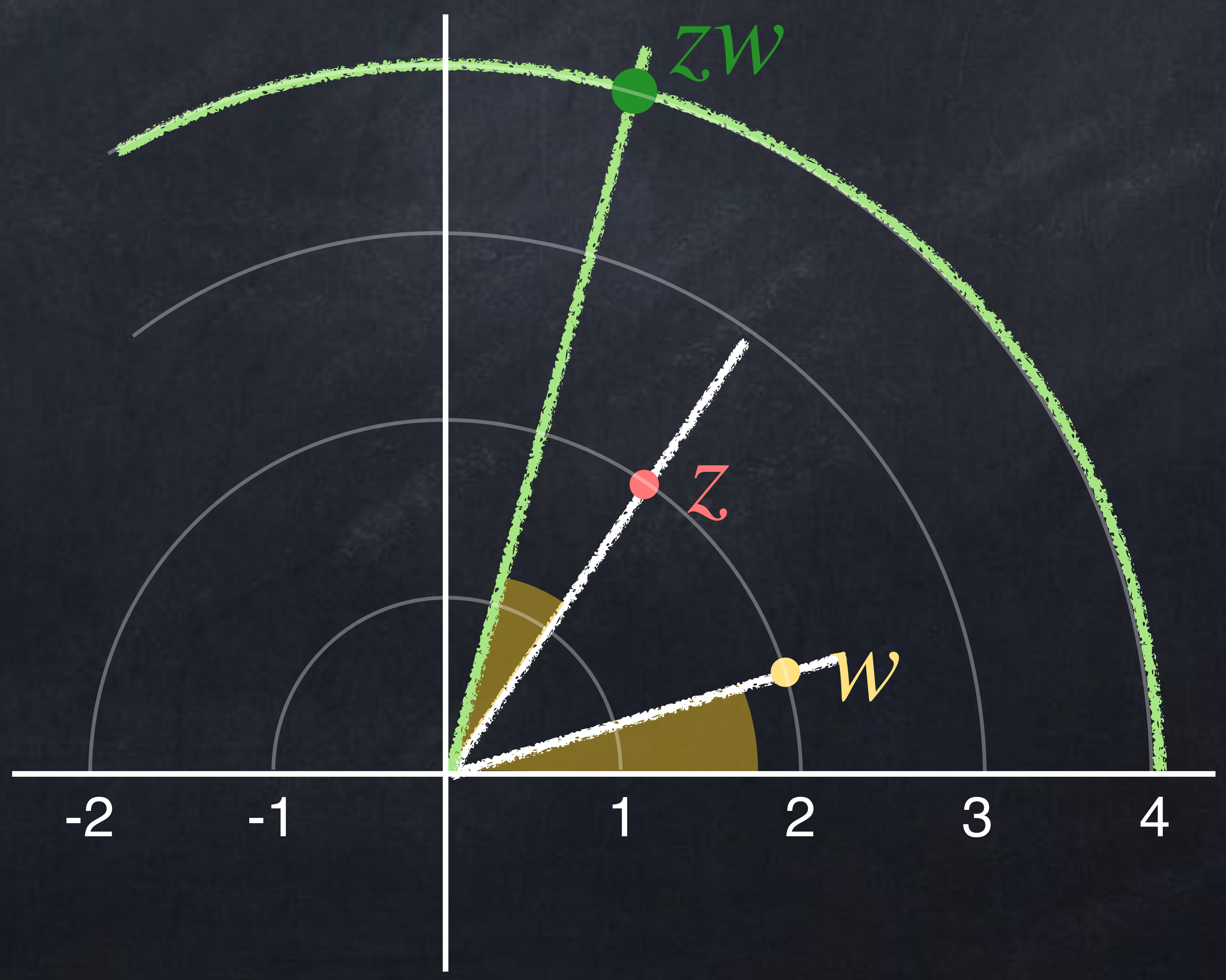
Multiplication

In general, $(re^{\theta i}) \cdot (se^{\phi i}) = (rs)e^{(\theta+\phi)i}$.

What does this mean *visually*? multiply magnitudes, add angles

Let's draw a dot • at zw below.





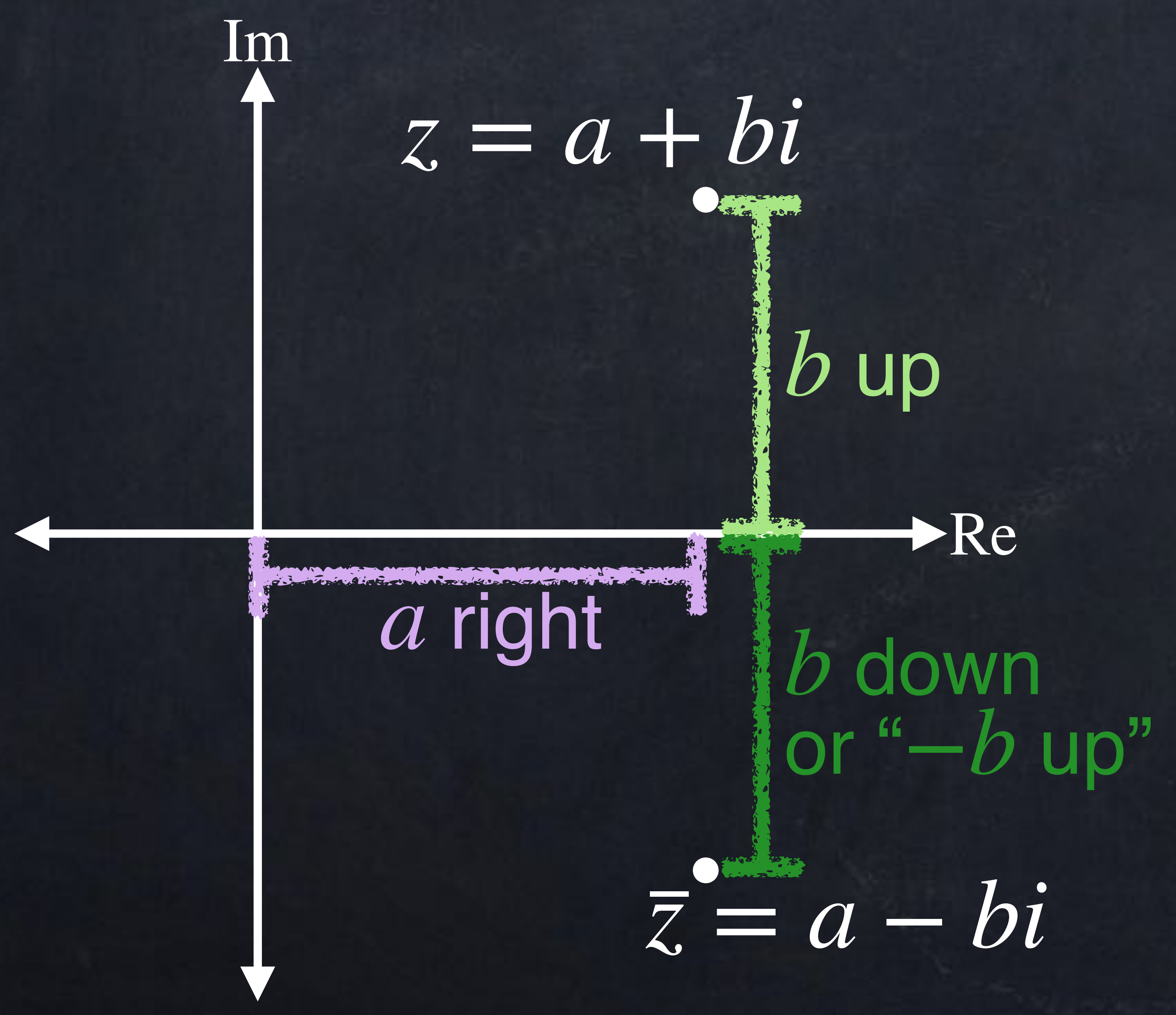
Complex conjugate

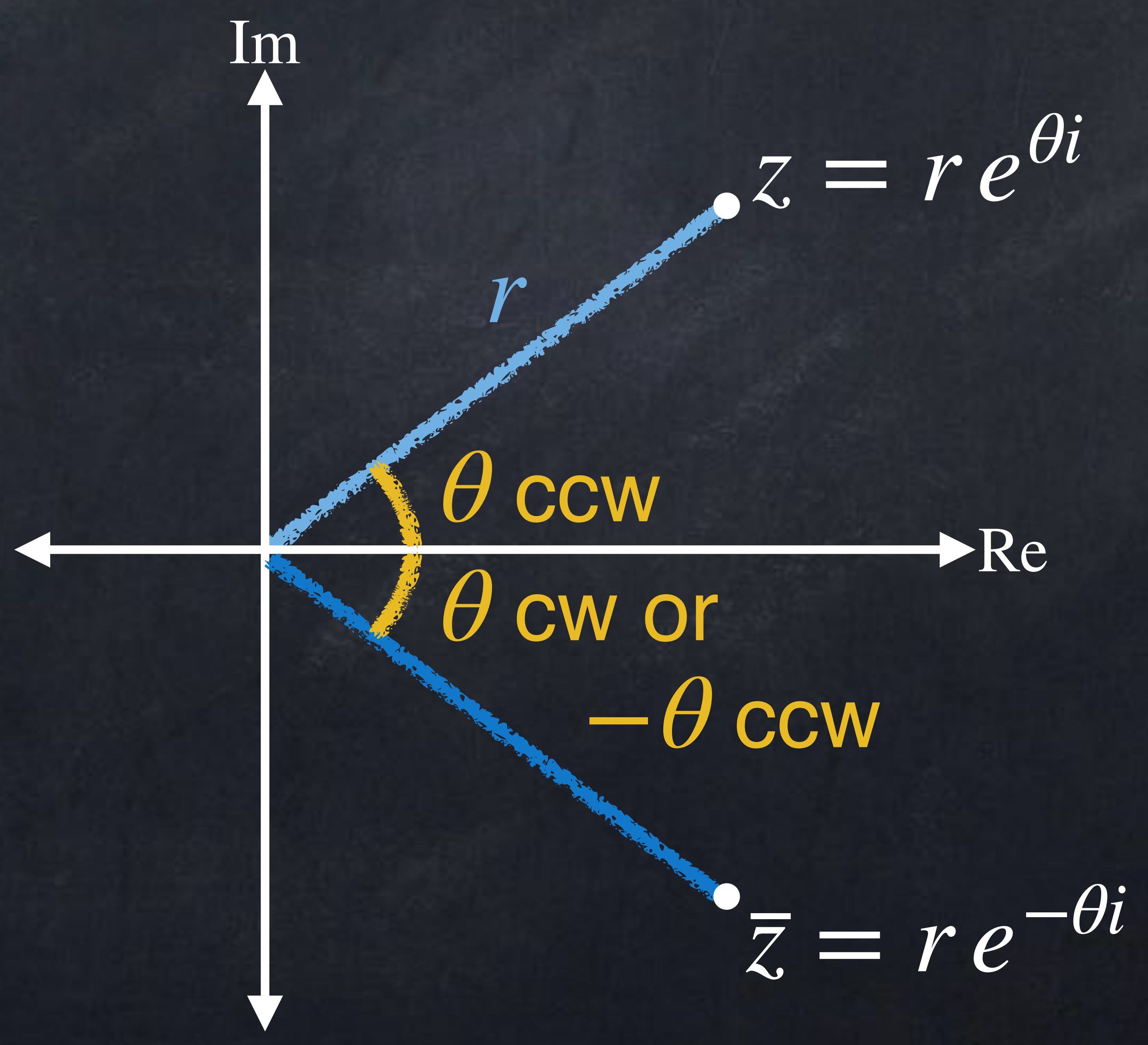
Last time

How is $\overline{z}$ calculated?

$$\overline{a + bi} = a - bi$$

$$\overline{r e^{\theta i}} = r e^{-\theta i}$$





Task: For $z = 5\sqrt{3} - 5i$ and $w = 5e^{(\pi/6)i} = 5e^{(30^\circ)i}$, calculate $z \cdot \overline{w}$.

How easy it is to do calculations in rectangular form vs. exponential form?

		$a+bi$	$re^{\phi i}$
real part			
imaginary part			
magnitude	$\|z\|$		
argument	$\arg(z)$		
conjugate	$\bar{z}$		
sum/difference	$z \pm w$		
product	zw		
quotient	z/w		
powers	z^n		

Roots of unity

What numbers z satisfy $z^2 = 1$?

Answer: 1, -1

What are all the complex numbers z that satisfy $z^4 = 1$?

Note that $z^2 = 1$ or $z^2 = -1$.

Answer: 1, -1, i, -i

Roots of unity

What are all the complex numbers z that satisfy $z^3 = 1$?

Roots of unity

For any natural number n, the solutions to $z^n = 1$ are exactly

- $z = e^{(2\pi/n)i}$ ← Call this ω.
- $z = e^{2\cdot(2\pi/n)i} = \omega^2$
- $z = e^{3\cdot(2\pi/n)i} = \omega^3$

  $\vdots$
- $z = e^{(n-1)\cdot(2\pi/n)i} = \omega^{n-1}$
- $z = e^{n\cdot(2\pi/n)i} = \omega^n = 1.$

These are called the ***n*****th roots of unity**.

Real vs. complex

In some ways, real numbers are better:

- Physical measurements
- Ordered: always $x < y$ or $x \geq y$

In some ways, complex #s are better:

- n^{th} roots – always exactly n of them
- Rotation and trig functions
- Polynomials — …

The Fundamental Theorem of Algebra (ver. 1)

For any non-constant polynomial $f(x)$, there is at least one complex solution to $f(x) = 0$.

Not true for real (example: $x^2 + 1 = 0$).

Polynomials

A **polynomial** in the variable x is a function of real numbers that *can* be described by an expression of the form

😀$x^n +$ 🧐$x^{n-1} + \cdots +$ 😂$x^2 +$ 🥺$x +$ 😐,

where $n \geq 0$ is an integer and the emoji are real or complex numbers (called the **coefficients**).

A **real polynomial** is one where every coefficient is a real number.

A **complex polynomial** is one where every coefficient is complex.

- Real numbers are complex numbers ($a + 0i$), so every real polynomial is also a complex polynomial.

Examples of polynomials:

- $5x^3 - 27x + \frac{3}{2}$
- $\sqrt{82}x^5 - 9x$
- $(x-1)^3$ ⟵ This can be written as x^3+3x^2+3x+1, so it is a polynomial.
- 12
- $ax + b$ if the variable is x
- $7t^2 - 8t + 1$ if the variable is t

Examples that are *not* polynomials:

- $x^{1/2}$
- $5x^2 + 3 + x^{-1}$
- $\cos(x)$

Term and Degree

The **terms** of a polynomial are the expressions that are added together or subtracted.

- Example: $f(x) = x^5 + 6x^3 - 4x + 8$ has four terms:
  - x^5 is the “first term” and the “X to the fifth term”
  - $6x^3$ is the “second term” and the “X cubed term”
  - $-4x$ is the “third term” and the “X term”
  - 8 is the “fourth term” and the “constant term”

The order of the terms doesn’t matter.
Some people prefer to write $8 - 4x + 6x^3 + x^5$.

Term and Degree

The **degree** of a polynomial is the highest power of the variable that appears in the polynomial. We write $\deg(f)$ for the degree of $f(x)$.

- Degree 0 example: 9 — "constant"
- Degree 1 example: $x+2$ — "linear"*
- Degree 2 example: $2x^2-5x-12$ — "quadratic"
- Degree 3 example: $-8x^3$ — "cubic"
- Degree 4 example: x^4-7x+1 — "quartic"

A polynomial is called **monic** if its highest-degree term has coefficient 1.

- Example: $x^3-\frac{2}{5}x+8$

+ − × ÷

We can add two polynomials.

$$(4x^2 - 3x) + (x^3 + x^2 + 3x + 8) \quad = \quad x^3 + 5x^2 + 8$$

We can subtract two polynomials.

$$(4x^2 - 3x) - (x^3 + x^2 + 3x + 8) \quad = \quad -x^3 + 3x^2 - 6x - 8$$

We can multiply two polynomials.

$$(4x^2 - 3x)(x^3 + x^2 + 3x + 8) \quad = \quad 4x^5 + x^4 + 9x^3 + 23x^2 - 24x$$

We can try to divide two polynomials, but sometimes the result is not a polynomial (for example, $1/x$ is not a polynomial).

+ − × ÷

Question: What can we say about $\deg(f+g)$ and $\deg(f\cdot g)$?

$$(4x^2-3x)+(x^3+x^2+3x+8) \;=\; x^3+5x^2+8$$

$$(4x^2-3x)+(-4x^2+7) \;=\; -3x+7$$

$\deg(f+g)$ is $\leq$ the maximum of $\deg(f)$ and $\deg(g)$.

$$(4x^2-3x)(x^3+x^2+3x+8) \;=\; 4x^5+x^4+9x^3+23x^2-24x$$

$$= 4x^2(x^3+x^2+3x+8) + (-3x)(x^3+x^2+3x+8)$$

$$= (4x^5+\cdots) + (-3x^4+\cdots)$$

$$x^a\cdot x^b = x^{a+b}$$

$\deg(f\cdot g) = \deg(f)+\deg(g)$ exactly.

Roots or zeros

The number c is a **zero** of the polynomial f if $f(c) = 0$. A zero of a polynomial is also called a **root** of the polynomial.

Sometimes we are interested in particular types of numbers as zeros.

- Example: $2x^6 - 3x^5 - 21x^4 + 56x^3 - 26x^2 - 245x + 525$ has
  - Integer root: -3
  - Rational roots: -3 and $\frac{5}{2}$
  - Real roots: $-3, \frac{5}{2}, \sqrt{7},$ and $-\sqrt{7}$

    These are real numbers that are also rational.
  - Complex roots: $-3, \frac{5}{2}, \sqrt{7}, -\sqrt{7}, 1+2i,$ and $1-2i$

    These are complex numbers that are also real.

Roots or zeros

The number c is a **zero** of the polynomial f if $f(c) = 0$. A zero of a polynomial is also called a **root** of the polynomial.

We often use the variable z when we care about complex roots. For example,

- "What are the zeros of $x^2 + 1$?"

  Depending who you ask, the answer could be either "i and $-i$" or "there are no zeros".

- "What are the zeros of $z^2 + 1$?"

  Answer: i and $-i$.

Roots or zeros

The number c is a **zero** of the polynomial f if $f(c) = 0$. A zero of a polynomial is also called a **root** of the polynomial.

The Fundamental Theorem of Algebra (ver. 1)

Every non-constant complex polynomial has at least one root.